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Partially reversible quantum operations and their information-theoretical properties

Masashi Ban

Advanced Research Laboratory, Hitachi, Ltd, Akanuma 2520, Hatoyama, Saitama 350-0395, Japan

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Abstract

Partial reversibility of quantum operations (or quantum channels) is considered from the information-theoretical point of view. The necessary and sufficient condition for quantum operations to be partially reversible is shown. The condition can be expressed in terms of information-theoretical quantities (von Neumann entropy and Ψ-information). The quantum information-theoretical meanings of the condition are discussed. The results are compared with those obtained for completely reversible quantum operations. The Ψ-information is calculated for the quantum depolarizing channel of a qubit and the linear dissipative channel of a single-mode bosonic system.

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1. Introduction

Quantum operations (or quantum channels) are the most general transformations that map a quantum state into another, and they are mathematically represented by completely positive maps [1–4]. Quantum operations include dynamical evolution of quantum states in reversible and irreversible processes, state-transmission through quantum communication channels and state-change due to the effect of quantum measurement performed on systems. Unitary operations have the important property that there are physically realizable inverse operations. In general, quantum operations do not necessarily have physically realizable inverse. Here the physical realizability of inverse operations is very important. Suppose some relaxation process caused by an interaction with a thermal reservoir (or an environmental system), described by the time-evolution operator (quantum operation) $\hat{\mathcal{K}}(t) = \mathrm{e}^{t\hat{\Lambda}}$ which is derived by means of the nonequilibrium statistical mechanical method [5–7], where the quantum operation $\hat{\Lambda}$ is called the damping operator. Although such a time-evolution operator seems to have its inverse $\hat{\mathcal{K}}(-t) = \mathrm{e}^{-t\hat{\Lambda}}$, the inverse operation $\hat{\mathcal{K}}(-t)$ is clearly unphysical because of the second law of thermodynamics. It is an important problem in quantum information processing whether a given quantum operation is physically reversible or not. For example, in order that the quantum error correction is possible, the quantum operation which causes the errors must be reversible. The necessary and sufficient condition for quantum operations to be completely reversible is shown in [8–10]. The algebraic and information-theoretic properties of completely reversible quantum operations have been investigated in detail. The complete reversibility of a quantum operation $\hat{\mathcal{L}}$ means that for any quantum quantum state defined on a given Hilbert space, there exists a physically realizable quantum operation $\hat{\mathcal{R}}$ such that $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho} = \hat{\rho}$. The quantum operation $\hat{\mathcal{R}}$ is the left inverse of $\hat{\mathcal{L}}$. In quantum information processing, the quantum operation $\hat{\mathcal{R}}$ describes the quantum error correcting process. The complete reversibility imposes a tight condition on quantum operations. In this paper, we introduce a partial reversibility of quantum operations. The partial reversibility of quantum operations means that for quantum states $\hat{\rho}$ satisfying some condition, there exists a physically realizable quantum operation $\hat{\mathcal{R}}$ such that $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho} = \hat{\rho}$. In this case, the existence of the quantum operation $\hat{\mathcal{R}}$ depends on the property of the quantum state $\hat{\rho}$. It is obvious that complete reversible quantum operations are partially reversible. We will show the necessary and sufficient condition for quantum operations to be partially reversible, and we will investigate the properties of partially reversible quantum operations.

In section 2, we briefly summarize the complete reversibility of quantum operations. The necessary and sufficient condition for the complete reversibility of quantum operations is explained. In section 3, we introduce the partial reversibility of quantum operations, and we obtain the necessary and sufficient condition for quantum operations to be partially reversible. For this purpose, we introduce Ψ-entropy and Ψ-information which correspond to the entropy exchange and the coherent information in the complete reversibility of quantum operations. Furthermore, we investigate the basic properties of partially reversible quantum operations. In section 4, we calculate the Ψ-information for the quantum depolarizing channel of a qubit and the linear dissipative channel of a single-mode bosonic system. We compare the results with those obtained for coherent information. In section 5, a summary is given.

2. Completely reversible quantum operations

In this section, we briefly review the necessary and sufficient condition for the complete reversibility of quantum operations that is closely related to the quantum error correction [9, 10]. A quantum operation $\hat{\mathcal{L}}$ is a completely positive map which transforms one quantum state to another in the most general way. Suppose a quantum state $\hat{\rho}$ whose support space is an N-dimensional Hilbert space $\mathcal{H}_N$. Then the eigenvectors $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle\}$ of the quantum state $\hat{\rho}$ become a complete orthonormal system of the Hilbert space $\mathcal{H}_N$. A trace-preserving quantum operation $\hat{\mathcal{L}}$ acting on quantum states of the Hilbert space $\mathcal{H}_N$ can be expressed in two equivalent forms [2]. One is called the operator-sum representation (or the Kraus representation)

$$\hat{\mathcal{L}}\hat{\rho} = \sum_{\mu} \hat{A}_{\mu}\hat{\rho}\hat{A}_{\mu}^{\dagger} \tag{1}$$

with

$$\sum_{\mu} \hat{A}_{\mu}^{\dagger}\hat{A}_{\mu} = \hat{1} \tag{2}$$

where $\hat{A}_{\mu}$ is an operator defined on the Hilbert space $\mathcal{H}_N$. The other is called the unitary representation

$$\hat{\mathcal{L}}\hat{\rho} = \mathrm{Tr}_E[\hat{U}(\hat{\rho} \otimes |0^E\rangle\langle 0^E|)\hat{U}^{\dagger}] \tag{3}$$

where $\hat{U}$ is the unitary operator which describes the state change caused by the interaction with an appropriate environmental system, the initial state of which is $|0^E\rangle$ and Tr_E stands for the partial trace over the Hilbert space $\mathcal{H}^E$ of the environmental system.

A quantum operation $\hat{\mathcal{L}}$ is called completely reversible if there exists a physically realizable operation $\hat{\mathcal{R}}$ such that $\hat{\mathcal{R}}\hat{\mathcal{L}} = \hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is an identity operation acting on operators of the Hilbert space $\mathcal{H}_N$. The necessary and sufficient condition for a quantum operation $\hat{\mathcal{L}}$ to be completely reversible is given by [10]

$$\langle\psi_j|\hat{A}_\nu^\dagger\hat{A}_\mu|\psi_k\rangle = M_{\mu\nu}\delta_{jk} \tag{4}$$

where $\mathbf{M} = (M_{\mu\nu})$ is a Hermitian matrix which does not depend on either j or k. Of course, any unitary operation satisfies this condition. For unitary operations, $\mathbf{M}$ becomes a unit matrix. Condition (4) is equivalent to the following information-theoretical relation [9]

$$S(\hat{\rho}) = I_C(\hat{\rho}, \hat{\mathcal{L}}) \tag{5}$$

where $S(\hat{\rho}) = -\mathrm{Tr}[\hat{\rho}\log\hat{\rho}]$ is the von Neumann entropy of a quantum state $\hat{\rho}$ and $I_C(\hat{\rho}, \hat{\mathcal{L}})$ is the coherent information of a quantum operation $\hat{\mathcal{L}}$. The coherent information plays a similar role in quantum information theory as the Shannon mutual information does in classical information theory. The coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ is defined by

$$I_C(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\mathcal{L}}\hat{\rho}) - S_e(\hat{\rho}, \hat{\mathcal{L}}) \tag{6}$$

with

$$S_e(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}}|\Psi\rangle\langle\Psi|) \tag{7}$$

where $|\Psi\rangle$ is a purification of a quantum state $\hat{\rho}$, the partial trace of which becomes $\hat{\rho}$, and $S_e(\hat{\rho}, \hat{\mathcal{L}})$ is called the entropy exchange of a quantum operation $\hat{\mathcal{L}}$. Although relation (5) seems to depend on the spectrum of the quantum state $\hat{\rho}$, it depends only on the quantum operation $\hat{\mathcal{L}}$ and the support space $\mathcal{H}_N$ of the quantum state $\hat{\rho}$. In fact, if relation (5) is satisfied for some quantum state having the support space $\mathcal{H}_N$, it holds for all quantum states defined on the support space $\mathcal{H}_N$.

The coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ can take negative values and satisfies the inequality

$$-S(\hat{\rho}) \leqslant I_C(\hat{\rho}, \hat{\mathcal{L}}) \leqslant S(\hat{\rho}). \tag{8}$$

Furthermore, the coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ satisfies the quantum data processing inequality [8, 9]. For trace-preserving quantum operations $\hat{\mathcal{L}}$ and $\hat{\mathcal{K}}$, the quantum data processing inequality is given by

$$I_C(\hat{\rho}, \hat{\mathcal{L}}) \geqslant I_C(\hat{\rho}, \hat{\mathcal{K}}\hat{\mathcal{L}}). \tag{9}$$

Note that the coherent information does not necessarily satisfy the inequality $I_C(\hat{\rho}, \hat{\mathcal{K}}) \geqslant I_C(\hat{\rho}, \hat{\mathcal{K}}\hat{\mathcal{L}})$. The entropy exchange $S_e(\hat{\rho}, \hat{\mathcal{L}})$ satisfies the quantum Fano inequality [8, 9]

$$S_e(\hat{\rho}, \hat{\mathcal{L}}) \leqslant H[F_e(\hat{\rho}, \hat{\mathcal{L}})] + [1 - F_e(\hat{\rho}, \hat{\mathcal{L}})]\log(N^2 - 1) \tag{10}$$

where $H(x) = -x\log x - (1-x)\log(1-x)$ and $F_e(\hat{\rho}, \hat{\mathcal{L}})$ is the entanglement fidelity defined by

$$\begin{aligned} F_e(\hat{\rho}, \hat{\mathcal{L}}) &= \langle\Psi|(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}}|\Psi\rangle\langle\Psi|)|\Psi\rangle \\ &= \sum_\mu |\,\mathrm{Tr}(\hat{A}_\mu\hat{\rho})|^2 \end{aligned} \tag{11}$$

with $|\Psi\rangle$ being a purification of the quantum state $\hat{\rho}$. The entropy exchange plays a similar role in the quantum information theory as the conditional entropy does in the classical information theory. It is obvious from the quantum Fano inequality that the entropy exchange $S_e(\hat{\rho}, \hat{\mathcal{L}})$ vanishes if $F_e(\hat{\rho}, \hat{\mathcal{L}}) = 1$. Note that in the classical information theory, the conditional entropy becomes zero if the average probability of error is zero [11]. The complete reversibility of a quantum operation $\hat{\mathcal{L}}$ is equivalent to the fact that there is a quantum operation $\hat{\mathcal{R}}$ such that $F_e(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 1$.

3. Partially reversible quantum operations

3.1. Partial reversibility of quantum operations

The number of complete orthonormal systems in an N-dimensional Hilbert space $\mathcal{H}_N$ is infinite. Any unitary transformation maps one complete orthonormal system into another. Hence we denote a complete orthonormal system $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle\}$ of the Hilbert space $\mathcal{H}_N$ as Ψ and a family of all possible complete orthonormal systems as $\mathcal{S} = \{\Psi_1, \Psi_2, \ldots\}$. Furthermore, we denote a set of quantum states as $\mathcal{Q}(\Psi)$, such that the eigenstates of the quantum states $\hat{\rho} \in \mathcal{Q}(\Psi)$ belong to the complete orthonormal system Ψ. Then, using this notation, we can say that for a completely reversible quantum operation $\hat{\mathcal{L}}$ satisfying conditions (4) and (5), there exists a quantum operation $\hat{\mathcal{R}}$ such that $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho} = \hat{\rho}$ for $\forall\hat{\rho} \in \mathcal{Q}(\Psi)$ with $\forall\Psi \in \mathcal{S}$. We now introduce a partial reversibility of quantum operations as follows.

Definition 1. *A quantum operation $\hat{\mathcal{L}}$ is partially reversible if a left inverse quantum operation $\hat{\mathcal{R}}$ exists for all quantum states $\hat{\rho}$ that have their eigenstates in some complete orthonormal system $\Psi = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle\}$, that is, $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho} = \hat{\rho}$ for $\forall\hat{\rho} \in \mathcal{Q}(\Psi)$ with $\exists\Psi \in \mathcal{S}$.*

Remarks. The partial reversibility of quantum operations depends on which complete orthonormal system Ψ is chosen. It is obvious from the definition that completely reversible quantum operations are partially reversible. The partial reversibility of quantum operation is closely related to a classical correlation of bipartite quantum states while the complete reversibility is related to a quantum entanglement (see section 3.4). In this section, we consider the basic properties of partially reversible quantum operations.

To clarify the definition of the partial reversibility, let $\hat{\mathcal{L}}$ be a partially reversible quantum operation and $\hat{\mathcal{R}}$ its reversing operation. Then the equality $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\mathcal{J}}\hat{X} = \hat{\mathcal{J}}\hat{X}$ holds for any operator $\hat{X}$ defined on the Hilbert space $\mathcal{H}_N$, where the completely positive map $\hat{\mathcal{J}}$ is given by $\hat{\mathcal{J}}\hat{X} = \sum_{k=1}^{N} \hat{P}_k\hat{X}\hat{P}_k$ with the orthogonal projector $\hat{P}_k = |\psi_k\rangle\langle\psi_k|$. Using the operator-sum representation, $\hat{\mathcal{L}}\hat{X} = \sum_\mu \hat{A}_\mu\hat{X}\hat{A}_\mu^\dagger$ and $\hat{\mathcal{R}}\hat{X} = \sum_\xi \hat{R}_\xi\hat{X}\hat{R}_\xi^\dagger$, we obtain the relation

$$\sum_\xi\sum_\mu\sum_k \hat{B}_{\xi\mu k}\hat{X}\hat{B}_{\xi\mu k}^\dagger = \sum_k \hat{P}_k\hat{X}\hat{P}_k \tag{12}$$

where $\hat{B}_{\xi\mu k} = \hat{R}_\xi\hat{A}_\mu\hat{P}_k$. Since both sides represent the same completely positive map, there are parameters $N_{\xi\mu}^{kl}$ which satisfy the relation $\hat{B}_{\xi\mu k} = \sum_l N_{\xi\mu}^{kl}\hat{P}_l$ [4]. Using the fact that the quantum operations $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ are trace preserving and $\hat{P}_k\hat{P}_l = \delta_{kl}\hat{P}_k$, we find that the parameter $N_{\xi\mu}^{kl}$ satisfies

$$N_{\xi\mu}^{kl} = \delta_{kl}N_{\xi\mu}^{k} \qquad \sum_\xi\sum_\mu \left|N_{\xi\mu}^{k}\right|^2 = 1. \tag{13}$$

Then the operator $\hat{A}_\nu^\dagger\hat{A}_\mu$ is calculated to be

$$\begin{aligned}\hat{A}_\nu^\dagger\hat{A}_\mu &= \sum_k \hat{P}_k\hat{A}_\nu^\dagger\left(\sum_\xi \hat{R}_\xi^\dagger\hat{R}_\xi\right)\hat{A}_\mu\sum_{l=1}\hat{P}_l \\ &= \sum_\xi\sum_k\sum_l \hat{B}_{\xi\nu k}^\dagger\hat{B}_{\xi\mu l} \\ &= \sum_k M_{\mu\nu}(k)\hat{P}_k\end{aligned} \tag{14}$$

with

$$M_{\mu\nu}(k) = \sum_{\xi} N^{k}_{\xi\mu} N^{k*}_{\xi\nu} \tag{15}$$

which satisfies $\sum_{\mu} M_{\mu\mu}(k) = 1$ and $M_{\mu\mu}(k) \geqslant 0$. The parameter $M_{\mu\nu}$ is equal to that appearing in equation (30). Thus when a polar decomposition of the operator $\hat{A}_{\mu}$ is expressed as $\hat{A}_{\mu} = \hat{V}_{\mu}|\hat{A}_{\mu}|$ with $\hat{V}_{\mu}$ being a partial isometric operator, $\hat{V}^{\dagger}_{\mu}\hat{V}_{\mu} = \hat{1}$ [4], the amplitude operator $|\hat{A}_{\mu}|$ is given by

$$|\hat{A}_{\mu}| = \sum_{k} \sqrt{M_{\mu\mu}(k)}\,\hat{P}_{k}. \tag{16}$$

This result means that the partially reversible quantum operation $\hat{\mathcal{L}}$ is expressed as the canonical form. When the same argument is applied to completely reversible quantum operations, we can find the relation $|\hat{A}_{\mu}| = \sqrt{M_{\mu\mu}}\,\hat{1}$ for completely reversible quantum operations with operator-sum representation $\hat{\mathcal{L}}\hat{X} = \sum_{\mu} \hat{A}_{\mu}\hat{X}\hat{A}^{\dagger}_{\mu}$, where $M_{\mu\nu}$ is the same parameter that appeared in equation (4).

As a simple example, we consider the phase-erasing quantum operation given by

$$\hat{\mathcal{L}}_{\mathrm{p}}\hat{\rho} = \left(1 - \tfrac{1}{2}p\right)\hat{\rho} + \tfrac{1}{2}p\hat{\sigma}_{z}\hat{\rho}\hat{\sigma}_{z} \tag{17}$$

where $\hat{\sigma}_{z}$ is the z-component of the Pauli spin matrix and $\hat{\rho}$ is a quantum state defined on a two-dimensional Hilbert space. It is easy to see that the equality $\hat{\mathcal{L}}_{\mathrm{p}}\hat{\rho} = \hat{\rho}$ holds for any quantum state $\hat{\rho}$ that is diagonal with respect to the eigenstates of $\hat{\sigma}_{z}$ since the quantum operation $\mathcal{L}_{\mathrm{p}}$ yields the relation

$$\mathcal{L}_{\mathrm{p}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & (1-p)b \\ (1-p)c & d \end{pmatrix}. \tag{18}$$

Thus the quantum operation $\hat{\mathcal{L}}_{\mathrm{p}}$ given by equation (17) is partially reversible, where the reversing operation is an identity, that is, $\hat{\mathcal{R}} = \hat{\mathcal{I}}$. The operator-sum representation of $\hat{\mathcal{L}}_{\mathrm{p}}$ is given by $\hat{\mathcal{L}}_{\mathrm{p}}\hat{\rho} = \sum_{\mu=0}^{1} \hat{A}_{\mu}\hat{\rho}\hat{A}^{\dagger}_{\mu}$ with $\hat{A}_{0} = \sqrt{1 - \frac{1}{2}p}\,\hat{1}$ and $\hat{A}_{1} = \sqrt{\frac{1}{2}p}\,\hat{\sigma}_{z}$, and thus the parameters $M_{\mu\nu}(k)$ become

$$M_{00}(0) = M_{00}(1) = 1 - \tfrac{1}{2}p \qquad M_{11}(0) = M_{11}(1) = \tfrac{1}{2}p \tag{19}$$

$$M_{01}(0) = M_{10}(0) = -M_{01}(1) = -M_{10}(1) = \sqrt{\tfrac{1}{2}p\left(1 - \tfrac{1}{2}p\right)} \tag{20}$$

where $\hat{P}_{0} = |0\rangle\langle 0|$ and $\hat{P}_{1} = |1\rangle\langle 1|$ with $\hat{\sigma}_{z}|0\rangle = |0\rangle$ and $\hat{\sigma}_{z}|1\rangle = -|1\rangle$. In this case, the partial isometric operators $\hat{V}_{\mu}$ which satisfy the relation $\hat{A}_{\mu} = \hat{V}_{\mu}|\hat{A}_{\mu}|$ become unitary, that is, $\hat{V}_{0} = \hat{1}$ and $\hat{V}_{\mu} = \hat{\sigma}_{z}$.

3.2. The Ψ-entropy and Ψ-information

To investigate the properties of partially reversible quantum operations, we introduce several information-theoretic quantities. The first is Ψ-fidelity.

Definition 2. *For a quantum operation $\hat{\mathcal{L}}$, Ψ-fidelity $F_{\Psi}(\hat{\rho}, \hat{\mathcal{L}})$ of a quantum state $\hat{\rho} \in \mathcal{Q}(\Psi)$ which has the spectral decomposition $\hat{\rho} = \sum_{k=1}^{N} \pi_{k}|\psi_{k}\rangle\langle\psi_{k}|$ ($|\psi_{k}\rangle \in \Psi$) is defined by the average value of the fidelity between quantum states $|\psi_{k}\rangle\langle\psi_{k}|$ and $\hat{\mathcal{L}}|\psi_{k}\rangle\langle\psi_{k}|$ with the probability π_{k}*

$$F_{\Psi}(\hat{\rho}, \hat{\mathcal{L}}) = \sum_{k=1}^{N} \pi_{k}\langle\psi_{k}|(\hat{\mathcal{L}}|\psi_{k}\rangle\langle\psi_{k}|)|\psi_{k}\rangle \tag{21}$$

where the probability π_{k} is the eigenvalue of the quantum state $\hat{\rho}$.

Remarks. It is seen from the definitions, (11) and (21), that the equality $F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 1$ means $\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k| = |\psi_k\rangle\langle\psi_k|$ $(k = 1, 2, \ldots, N)$ while the equality $F_e(\hat{\rho}, \hat{\mathcal{L}}) = 1$ indicates $\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_k| = |\psi_j\rangle\langle\psi_k|$ $(j, k = 1, 2, \ldots, N)$. Although the entanglement fidelity $F_e(\hat{\rho}, \hat{\mathcal{L}})$ is independent of the choice of the complete orthonormal system Ψ (see equation (11)), the Ψ-fidelity $F_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ does depend on the choice of the set Ψ. Then we obtain the following proposition.

Proposition 1. *The entanglement fidelity $F_e(\hat{\rho}, \hat{\mathcal{L}})$ is not greater than the Ψ-fidelity $F_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ and the inequality is satisfied*

$$0 \leqslant F_e(\hat{\rho}, \hat{\mathcal{L}}) \leqslant F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \leqslant 1. \tag{22}$$

It is obvious from this proposition that if $F_e(\hat{\rho}, \hat{\mathcal{L}}) = 1$, the equality $F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 1$ holds while the equality $F_e(\hat{\rho}, \hat{\mathcal{L}}) = 1$ is not necessarily satisfied even if $F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 1$. The partial reversibility of a quantum operation $\hat{\mathcal{L}}$ is equivalent to the fact that there is a quantum operation $\hat{\mathcal{R}}$ such that $F_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 1$. Note that the complete reversibility means the existence of $\hat{\mathcal{R}}$ such that $F_e(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 1$. The proof of proposition 1 is given in appendix A. We next introduce Ψ-entropy of a quantum state.

Definition 3. *For a quantum operation $\hat{\mathcal{L}}$, Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ of a quantum state $\hat{\rho} = \sum_{k=1}^N \pi_k|\psi_k\rangle\langle\psi_k|$ $(|\psi_k\rangle \in \Psi)$ is the average value of the von Neumann entropy $S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|)$ with the probability π_k, that is,*

$$\begin{aligned} S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) &= \sum_{k=1}^N \pi_k S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|) \\ &= -\sum_{k=1}^N \pi_k \operatorname{Tr}[(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|) \log(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|)]. \end{aligned} \tag{23}$$

Finally, we introduce Ψ-information of a quantum state.

Definition 4. *For a quantum operation $\hat{\mathcal{L}}$, Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ is defined by*

$$I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\mathcal{L}}\hat{\rho}) - S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \tag{24}$$

Remarks. The Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ plays the same role in the partial reversibility of quantum operations as the entropy exchange $S_e(\hat{\rho}, \hat{\mathcal{L}})$ given by equation (7) does in the complete reversibility. Moreover, the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ corresponds to the coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ given by equation (6). Information-theoretical quantities similar to equation (24) have been introduced in [12, 13]. For a quantum state $\hat{\rho} \in \mathcal{Q}(\Psi)$ whose spectral decomposition is $\hat{\rho} = \sum_k \pi_k|\psi_k\rangle\langle\psi_k|$, we introduce a separable state $\hat{W}_\rho$ by $\hat{W}_\rho = \sum_k \pi_k|\psi_k\rangle\langle\psi_k| \otimes |\psi_k\rangle\langle\psi_k|$. Then it is obvious that the partial trace of $\hat{W}_\rho$ is the quantum state $\hat{\rho}$. Then the average value $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ of the von Neumann entropy can be expressed as

$$S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}}\hat{W}_\rho) - S(\hat{\rho}). \tag{25}$$

The Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ and the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ have some information-theoretical properties. The following proposition is important in the consideration of the partial reversibility of quantum operations.

Proposition 2. *The Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ is non-negative and not greater than* $\min[S(\hat{\rho}), S(\hat{\mathcal{L}}\hat{\rho})]$,

$$0 \leqslant I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \leqslant \min[S(\hat{\rho}), S(\hat{\mathcal{L}}\hat{\rho})] \tag{26}$$

and it satisfies the quantum data processing inequality for trace-preserving quantum operations $\hat{\mathcal{L}}$ and $\hat{\mathcal{K}}$,

$$I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \geqslant I_\Psi(\hat{\rho}, \hat{\mathcal{K}}\hat{\mathcal{L}}). \tag{27}$$

Moreover, the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ satisfies the quantum Fano inequality

$$S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \leqslant H[F_\Psi(\hat{\rho}, \hat{\mathcal{L}})] + [1 - F_\Psi(\hat{\rho}, \hat{\mathcal{L}})] \log(N - 1) \tag{28}$$

where $H(x) = -x \log x - (1 - x) \log(1 - x)$.

Remarks. It is important to note that the inequality $I_\Psi(\hat{\rho}, \hat{\mathcal{K}}) \geqslant I_\Psi(\hat{\rho}, \hat{\mathcal{K}}\hat{\mathcal{L}})$ is not necessarily satisfied. If the equality $F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 1$ holds, the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ vanishes. The dimensionality N of the Hilbert space $\mathcal{H}_N$ appears in the quantum Fano inequality of $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ while N^2 appears in that of $S_e(\hat{\rho}, \hat{\mathcal{L}})$. The Ψ-entropy is related to the reversibility of a diagonal matrix and the entropy exchange is related to that of an arbitrary $N \times N$ matrix. The proof of the proposition is given in appendix B. These properties imply that the Ψ-information and the Ψ-entropy are similar to the Shannon mutual information and the conditional entropy in the classical information theory [11].

3.3. The necessary and sufficient condition for partial reversibility

As explained in section 2, a quantum operation $\hat{\mathcal{L}}$ becomes completely reversible if and only if the coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ of a quantum operation $\hat{\mathcal{L}}$ is equal to the von Neumann entropy $S(\hat{\rho})$ for any quantum state $\hat{\rho}$ whose support space is the Hilbert space $\mathcal{H}_N$. In the same way, we can obtain the following theorem.

Theorem 1. *A quantum operation $\hat{\mathcal{L}}$ becomes partially reversible with respect to the complete orthonormal system $\Psi = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle\}$ if and only if the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ is equal to the von Neumann entropy $S(\hat{\rho})$ for any quantum state $\hat{\rho}$ whose eigenstates belong to the set Ψ, that is,*

$$\hat{\mathcal{L}} \text{ is partially reversible} \quad \Longleftrightarrow \quad I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho}) \tag{29}$$

with $\hat{\rho} = \sum_{k=1}^{N} \lambda_k |\psi_k\rangle\langle\psi_k| \in \mathcal{Q}(\Psi)$.

Although the condition seems to depend on the spectrum of the quantum state $\hat{\rho}$, namely, λ_k $(k = 1, 2, \ldots, N)$, it is independent of the spectrum and depends only on the complete orthonormal system Ψ. We also obtain the equivalent theorem.

Theorem 2. *The information-theoretic condition (29) for partial reversibility is equivalent to the following algebraic condition:*

$$\hat{\mathcal{L}} \text{ is partially reversible} \quad \Longleftrightarrow \quad \langle\psi_j|\hat{A}_\nu^\dagger \hat{A}_\mu|\psi_k\rangle = M_{\mu\nu}(j)\delta_{jk} \tag{30}$$

where $\hat{\mathcal{L}}\hat{\rho} = \sum_\mu \hat{A}_\mu \hat{\rho} \hat{A}_\mu^\dagger$ and $\mathbf{M}(j) = [M_{\mu\nu}(j)]$ is a Hermitian matrix which depends on the index j of the eigenstate $|\psi_k\rangle$ of $\hat{\rho}$.

Note that $M_{\mu\nu}(j)$ is equivalent to that given by equation (15). The proof of theorems 1 and 2 is given in appendix C.

If the Hermitian matrix $\mathbf{M}(j)$ appearing in equation (30) is independent of j, the partially reversible quantum operation $\hat{\mathcal{L}}$ becomes completely reversible. It is reasonable to consider

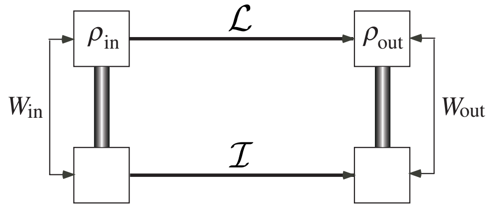

Figure 1. Schematic representation of the composite system and the quantum channel.

that the necessary and sufficient condition for the partial reversibility of quantum operations is weaker than that for the complete reversibility. In fact, we can obtain the following proposition.

Proposition 3. *The coherent information* $I_C(\hat{\rho}, \hat{\mathcal{L}})$ *and the* Ψ*-information* $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ *satisfy the inequality*

$$I_C(\hat{\rho}, \hat{\mathcal{L}}) \leqslant I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \leqslant S(\hat{\rho}) \tag{31}$$

where the equality holds for $\hat{\mathcal{L}} = \hat{\mathcal{I}}$.

It is obvious from this inequality that the equality $I_C(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ implies $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ and thus completely reversible quantum operations are always partially reversible. Of course, a partially reversible quantum operation is not necessarily completely reversible. The proof of proposition 3 is given in appendix D. When the condition $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ is satisfied, there is a quantum operation $\hat{\mathcal{R}}$ such that $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\sigma} = \hat{\sigma}$ for quantum states $\hat{\sigma}$ which commutes with the quantum state $\hat{\rho}$. On the other hand, if the equality $I_C(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ holds, there is a quantum operation $\hat{\mathcal{R}}$ such that $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\sigma} = \hat{\sigma}$ for any quantum state $\hat{\sigma}$ defined on the support space $\mathcal{H}_N$ of the quantum state $\hat{\rho}$.

3.4. Correlations in the reversibility of quantum operations

We consider correlations of quantum states under the influence of the reversible quantum operations. For any quantum state $\hat{\rho} \in \mathcal{Q}(\Psi)$, the spectral decomposition is expressed as $\hat{\rho} = \sum_{k=1}^{N} \pi_k |\psi_k\rangle\langle\psi_k|$ $(|\psi_k\rangle \in \Psi)$. We introduce a composite system in a quantum state $\hat{W}$, the reduced state of which becomes the quantum state $\hat{\rho}$. There are many quantum states $\hat{W}$ which satisfy this property. We consider here two typical composite quantum states. One is the entangled pure quantum state

$$\hat{W}^{(\mathrm{e})} = |\Psi\rangle\langle\Psi| \qquad |\Psi\rangle = \sum_{k=1}^{N} \sqrt{\pi_k}|\psi_k\rangle \otimes |\psi_k\rangle \tag{32}$$

and the other is the separable quantum state

$$\hat{W}^{(\mathrm{s})} = \sum_{k=1}^{N} \pi_k |\psi_k\rangle\langle\psi_k| \otimes |\psi_k\rangle\langle\psi_k|. \tag{33}$$

It is easy to see that the reduced quantum states of $\hat{W}^{(\mathrm{e})}$ and $\hat{W}^{(\mathrm{s})}$ become the quantum state $\hat{\rho}$. Suppose that one of the subsystems in the quantum state $\hat{W}^{(\mathrm{e})}$ or $\hat{W}^{(\mathrm{s})}$ is transmitted through a quantum channel $\hat{\mathcal{L}}$. Then the output quantum state of the composite system is given by $\hat{W}_{\text{out}} = (\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}_{\text{in}}$ with $\hat{W}_{\text{in}} = \hat{W}^{(\mathrm{e})}, \hat{W}^{(\mathrm{s})}$ (see figure 1).

The coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ is considered the measure of how much quantum entanglement can be transmitted through the quantum channel $\hat{\mathcal{L}}$ [9]. At the input side of the quantum channel $\hat{\mathcal{L}} \otimes \hat{\mathcal{I}}$, the entanglement of the composite system in the pure quantum state $\hat{W}^{(\mathrm{e})} = |\Psi\rangle\langle\Psi|$ is measured by the entropy of entanglement $E(|\Psi\rangle)$ [14] which is the von Neumann entropy of the reduced quantum of $\hat{W}^{(\mathrm{e})}$, that is, $E(|\Psi\rangle) = S(\hat{\rho})$. Thus it is found from equation (5) that the complete reversibility of the quantum channel $\hat{\mathcal{L}}$ means that the quantum channel $\hat{\mathcal{L}}$ transmits the quantum entanglement of the composite system without any loss. It is shown that when the distance between two quantum states is measured by means of the quantum relative entropy, the quantum state $\hat{W}^{(\mathrm{s})}$ is the closest separable state to the entangled pure state $\hat{W}^{(\mathrm{e})}$ [15]. The separable state $\hat{W}$ that minimizes the quantum relative entropy $S(\hat{W}^{(\mathrm{e})} : \hat{W})$ is given by $\hat{W} = \hat{W}^{(\mathrm{s})}$, where $S(\hat{\sigma}_1 : \hat{\sigma}_2) = \mathrm{Tr}[\hat{\sigma}_1(\log\hat{\sigma}_1 - \log\hat{\sigma}_2)]$. Since the equality $S(\hat{W}^{(\mathrm{e})} : \hat{W}^{(\mathrm{s})}) = S(\hat{\rho})$ holds, the distance between $\hat{W}^{(\mathrm{e})}$ and $\hat{W}^{(\mathrm{s})}$ measures the entanglement of the pure quantum state $\hat{W}^{(\mathrm{e})}$.

When the composite system is in the separable state $\hat{W}^{(\mathrm{s})}$, the total correlation between the input and output states of the quantum channel $\hat{\mathcal{L}}$ can be measured by the von Neumann mutual information [12, 13]:

$$\mathcal{I}_T(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho}) + S(\hat{\mathcal{L}}\hat{\rho}) - S(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}}\hat{W}^{(\mathrm{s})}). \tag{34}$$

Since the von Neumann entropy of the quantum state $(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}^{(\mathrm{s})}$ is given by equation (25), we obtain the equality $\mathcal{I}_T(\hat{\rho}, \hat{\mathcal{L}}) = I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ from equation (24). It is important to note that the separable state contains only the classical correlation between the subsystems. Hence this result implies that the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ measures how much classical correlation (or classical information) represented by the complete orthonormal system Ψ can be transmitted through the quantum channel $\hat{\mathcal{L}}$. When we represent the classical information in terms of the vectors belonging to the set Ψ, $S(\hat{\rho}) = -\sum_k \pi_k \log \pi_k$ represents the amount of classical information of the system in the quantum state $\hat{\rho} = \sum_{k=1}^{N} \pi_k |\psi_k\rangle\langle\psi_k|$. Thus it is found from equation (29) that the partial reversibility of the quantum channel $\hat{\mathcal{L}}$ means that the classical information represented by the complete orthonormal system Ψ is transmitted through the quantum channel $\hat{\mathcal{L}}$ without any loss.

When the composite system is in the entangled state $\hat{W}^{(\mathrm{e})}$, the total correlation between the input and output quantum states is given by the von Neumann mutual information [16]

$$\begin{aligned} I_T(\hat{\rho}, \hat{\mathcal{L}}) &= S(\hat{\rho}) + S(\hat{\mathcal{L}}\hat{\rho}) - S(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}}\hat{W}^{(\mathrm{s})}) \\ &= S(\hat{\rho}) + I_C(\hat{\rho}, \hat{\mathcal{L}}). \end{aligned} \tag{35}$$

The distance $D(\hat{\rho}, \hat{\mathcal{L}})$ between the output quantum states $(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}^{(\mathrm{e})}$ and $(\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}^{(\mathrm{s})}$ is given by

$$\begin{aligned} D(\hat{\rho}, \hat{\mathcal{L}}) &= S((\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}^{(\mathrm{e})} : (\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})\hat{W}^{(\mathrm{s})}) \\ &= I_T(\hat{\rho}, \hat{\mathcal{L}}) - \mathcal{I}_T(\hat{\rho}, \hat{\mathcal{L}}) \\ &= S(\hat{\rho}) + I_C(\hat{\rho}, \hat{\mathcal{L}}) - I_\Psi(\hat{\rho}, \hat{\mathcal{L}}). \end{aligned} \tag{36}$$

Thus the decrease of the distance between the quantum states caused by the quantum channel $\hat{\mathcal{L}}$ is equal to the difference between the coherent information and the Ψ-information of the quantum channel $\hat{\mathcal{L}}$, that is,

$$\begin{aligned} \Delta D(\hat{\rho}, \hat{\mathcal{L}}) &= D(\hat{\rho}, \hat{\mathcal{I}}) - D(\hat{\rho}, \hat{\mathcal{L}}) \\ &= I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) - I_C(\hat{\rho}, \hat{\mathcal{L}}). \end{aligned} \tag{37}$$

Since the quantum relative entropy satisfies the inequality $S(\hat{\mathcal{L}}\hat{\sigma}_1 : \hat{\mathcal{L}}\hat{\sigma}_2) \leqslant S(\hat{\sigma}_1 : \hat{\sigma}_2)$, we obtain $\Delta D(\hat{\rho}, \hat{\mathcal{L}}) \geqslant 0$. This implies the inequality $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \geqslant I_C(\hat{\rho}, \hat{\mathcal{L}})$.

3.5. The classical correspondence

We consider the classical correspondence of the Ψ-entropy and the Ψ-information. For this purpose, we assume that classical information is represented by vectors $|\psi_k\rangle$ $(N = 1, 2, \ldots, N)$ belonging to the complete orthonormal system Ψ of the Hilbert space $\mathcal{H}_N$. In this case, the quantum state of the information source with the Shannon entropy $H(X) = -\sum_{k=1}^{N} P_X(k) \log P_X(k)$ is described by the density matrix $\hat{\rho}_X = \sum_{k=1}^{N} P_X(k)|\psi_k\rangle \langle\psi_k|$. The channel matrix $P_{YX}(j|k)$ that characterizes the classical channel yields the output probability $P_Y(j) = \sum_{k=1}^{N} P_{YX}(j|k)P_X(k)$, where the channel matrix is normalized as $\sum_{j=1}^{N} P_{YX}(j|k) = 1$. The output state of the classical-like quantum channel becomes $\hat{\rho}_Y = \sum_{k=1}^{N} P_Y(k)|\psi_k\rangle\langle\psi_k|$, where the entropy of the output state is $H(Y) = -\sum_{j=1}^{N} P_Y(j) \log P_Y(j)$. Then the completely positive map $\hat{\mathcal{L}}_{YX}$ that transforms the input state $\hat{\rho}_X$ into the output state $\hat{\rho}_Y$ is given by

$$\hat{\rho}_Y = \hat{\mathcal{L}}_{YX}\hat{\rho}_X = \sum_{j=1}^{N}\sum_{k=1}^{N} \hat{A}_{jk}\hat{\rho}_X\hat{A}_{jk}^{\dagger} \qquad \hat{A}_{jk} = \sqrt{P_{YX}(j|k)}|\psi_j\rangle\langle\psi_k| \tag{38}$$

where $\sum_{j=1}^{N}\sum_{k=1}^{N} \hat{A}_{jk}^{\dagger}\hat{A}_{jk} = 1$. It is easy to see that the entropy exchange $S_e(\hat{\rho}_X, \hat{\mathcal{L}}_{YX})$ and the coherent information $I_C(\hat{\rho}_X, \hat{\mathcal{L}}_{YX})$ of the quantum channel $\hat{\mathcal{L}}_{YX}$ become

$$S_e(\hat{\rho}_X, \hat{\mathcal{L}}_{YX}) = H(Y, X) \qquad I_C(\hat{\rho}_X, \hat{\mathcal{L}}_{YX}) = -H(X|Y) \tag{39}$$

where $H(X|Y) = H(Y, X) - H(Y)$ is the conditional entropy and the joint entropy $H(Y, X)$ is given by

$$H(Y, X) = -\sum_{j=1}^{N}\sum_{k=1}^{N} P_{YX}(j, k) \log P_{YX}(j, k) \tag{40}$$

with $P_{YX}(j, k) = P_{YX}(j|k)P_X(k)$. On the other hand, the Ψ-entropy $S_\Psi(\hat{\rho}_X, \hat{\mathcal{L}}_{YX})$ and the Ψ-information $I_\Psi(\hat{\rho}_X, \hat{\mathcal{L}}_{YX})$ are calculated to be

$$S_\Psi(\hat{\rho}_X, \hat{\mathcal{L}}_{YX}) = H(Y|X) \qquad I_\Psi(\hat{\rho}_X, \hat{\mathcal{L}}_{YX}) = H(Y : Y) \tag{41}$$

where $H(Y|X) = H(Y) - H(Y : X)$ is the conditional entropy and $H(Y : X)$ is the Shannon mutual information

$$\begin{aligned} H(Y : X) &= \sum_{j=1}^{N}\sum_{k=1}^{N} P_{YX}(j|k)P_X(k) \log\left[\frac{P_{YX}(j|k)}{\sum_{m=1}^{N} P_{YX}(j|m)P_X(m)}\right] \\ &= H(Y) - H(Y|X) \\ &= H(X) - H(X|Y) \\ &= H(X) + H(Y) - H(Y, X). \end{aligned} \tag{42}$$

Therefore, for the classical-like quantum channel $\hat{\mathcal{L}}_{YX}$, the coherent information takes the negative value and the Ψ-information becomes equal to the Shannon mutual information. The von Neumann mutual information defined by $I_T(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho}) + I_C(\hat{\rho}, \hat{\mathcal{L}})$ [16] is also equal to the Shannon mutual information.

4. The Ψ-information of the simple quantum channels

In this section, we calculate the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ of the quantum depolarizing channel of a qubit and the linear dissipative channel of a single-mode bosonic channel. We compare the results with those obtained for the coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$.

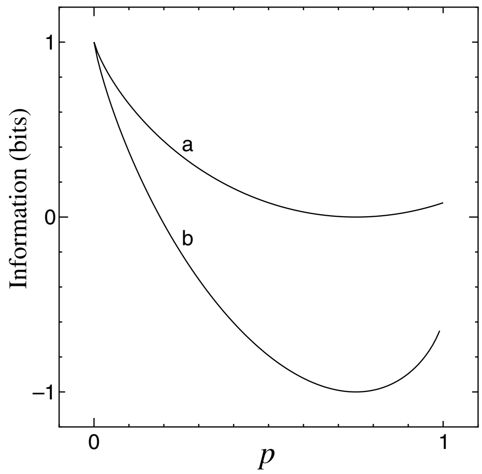

Figure 2. The Ψ-information (a) and the coherent information (b) of the quantum depolarizing channel of a qubit. In the figure, the information is measured in bits.

4.1. The quantum depolarizing channel of a qubit

The completely positive map $\hat{\mathcal{L}}$ that describes the quantum depolarizing channel of a qubit is given by

$$\begin{aligned}\hat{\mathcal{L}}\hat{\rho} &= (1-p)\hat{\rho} + \tfrac{1}{3}p(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + \hat{\sigma}_z\hat{\rho}\hat{\sigma}_z) \\ &= \left(1-\tfrac{4}{3}p\right)\hat{\rho} + \tfrac{2}{3}p\hat{1}\end{aligned} \tag{43}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the Pauli matrices. The complete positivity of the quantum channel $\hat{\mathcal{L}}$ imposes that the parameter p should be in the range $0 \leqslant p \leqslant 1$. We choose the complete orthonormal system $\Psi = \{|0\rangle, |1\rangle\}$, where $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$. For the sake of simplicity, we suppose here that the quantum state $\hat{\rho}$ is the statistical mixture of $|0\rangle$ and $|1\rangle$ with equal probabilities. Since we obtain

$$\hat{\mathcal{L}}|0\rangle\langle 0| = \left(1-\tfrac{2}{3}p\right)|0\rangle\langle 0| + \tfrac{2}{3}p|1\rangle\langle 1| \tag{44}$$

$$\hat{\mathcal{L}}|1\rangle\langle 1| = \tfrac{2}{3}p|0\rangle\langle 0| + \left(1-\tfrac{2}{3}p\right)|1\rangle\langle 1| \tag{45}$$

the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ becomes

$$S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = -\left(1-\tfrac{2}{3}p\right)\log\left(1-\tfrac{2}{3}p\right) - \left(\tfrac{2}{3}p\right)\log\left(\tfrac{2}{3}p\right) \tag{46}$$

and the von Neumann entropy $S(\hat{\mathcal{L}}\hat{\rho}) = \log 2$ is obtained. Hence the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ of the quantum depolarizing channel is given by

$$I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = \log 2 + \left(1-\tfrac{2}{3}p\right)\log\left(1-\tfrac{2}{3}p\right) + \left(\tfrac{2}{3}p\right)\log\left(\tfrac{2}{3}p\right). \tag{47}$$

On the other hand, the coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ of the quantum depolarizing channel is calculated to be

$$I_C(\hat{\rho}, \hat{\mathcal{L}}) = \log 2 + (1-p)\log(1-p) + p\log p - p\log 3. \tag{48}$$

The Ψ-information (47) and the coherent information (48) are plotted in figure 2. For the quantum depolarizing channel, the Ψ-fidelity $F_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ and the entanglement fidelity $F_e(\hat{\rho}, \hat{\mathcal{L}})$ are given respectively by

$$F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 1 - \tfrac{2}{3}p \qquad F_e(\hat{\rho}, \hat{\mathcal{L}}) = 1 - p. \tag{49}$$

Since $N = 2$, the quantum Fano inequality of the Ψ-entropy attains the equality.

4.2. The linear dissipative channel of a single-mode bosonic system

The completely positive map $\hat{\mathcal{L}}$ that describes the linear dissipative channel of a single-mode bosonic system is given by

$$\hat{\mathcal{L}} = \exp\left[g\left(\hat{\mathcal{K}}_+ - \hat{\mathcal{K}}_0 + \tfrac{1}{2}\right)\right] \qquad (g > 0) \tag{50}$$

where $\hat{\mathcal{K}}_\pm$ and $\hat{\mathcal{K}}_0$ form the $su(1,1)$ Lie algebra

$$\hat{\mathcal{K}}_+\hat{X} = \hat{a}^\dagger\hat{X}\hat{a} \qquad \hat{\mathcal{K}}_-\hat{X} = \hat{a}\hat{X}\hat{a}^\dagger \qquad \hat{\mathcal{K}}_0\hat{X} = \tfrac{1}{2}(\hat{a}^\dagger\hat{a}\hat{X} + \hat{X}\hat{a}^\dagger\hat{a} + \hat{X}) \tag{51}$$

with $\hat{a}$ and $\hat{a}^\dagger$ being bosonic annihilation and creation operators which satisfy the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{1}$. We suppose here that the quantum state $\hat{\rho}$ is given by

$$\hat{\rho} = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|\alpha\rangle\langle\alpha| \tag{52}$$

where $|0\rangle$ is the vacuum state and $|\alpha\rangle$ is the Glauber coherent state. Then the support space of the quantum state $\hat{\rho}$ is the two-dimensional Hilbert space $\mathcal{H}_2$ spanned by $|0\rangle$ and $|\alpha\rangle$. The output state of the quantum channel $\hat{\mathcal{L}}$ becomes

$$\hat{\mathcal{L}}\hat{\rho} = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|\alpha\,\mathrm{e}^{-\frac{1}{2}g}\rangle\langle\alpha\,\mathrm{e}^{-\frac{1}{2}g}|. \tag{53}$$

The von Neumann entropies of the input and output quantum states are calculated to be

$$S(\hat{\rho}) = G(\kappa) \qquad S(\hat{\mathcal{L}}\hat{\rho}) = G(\tilde{\kappa}) \tag{54}$$

where $\kappa = \mathrm{e}^{-(1/2)|\alpha|^2}$ and $\tilde{\kappa} = \mathrm{e}^{-(1/2)|\alpha|^2\mathrm{e}^{-g}}$. In this equation, the function $G(x)$ is defined by

$$G(x) = \log 2 - \tfrac{1}{2}(1+x)\log(1+x) - \tfrac{1}{2}(1-x)\log(1-x). \tag{55}$$

The spectral decomposition of the quantum state $\hat{\rho}$ is given by

$$\hat{\rho} = \tfrac{1}{2}(1+\kappa)|\psi_0\rangle\langle\psi_1| + \tfrac{1}{2}(1-\kappa)|\psi_1\rangle\langle\psi_1| \tag{56}$$

with

$$|\psi_0\rangle = \frac{|0\rangle + |\alpha\rangle}{\sqrt{2(1+\kappa)}} \qquad |\psi_1\rangle = \frac{|0\rangle - |\alpha\rangle}{\sqrt{2(1-\kappa)}}. \tag{57}$$

We can choose the complete orthonormal system $\Psi = \{|\psi_0\rangle, |\psi_1\rangle\}$ of the support space $\mathcal{H}_2$ of the quantum state $\hat{\rho}$. Then, after some calculation, we obtain the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$

$$\begin{aligned} S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) &= \tfrac{1}{2}(1+\kappa)S(\hat{\mathcal{L}}|\psi_0\rangle\langle\psi_0|) + \tfrac{1}{2}(1-\kappa)S(\hat{\mathcal{L}}|\psi_1\rangle\langle\psi_1|) \\ &= G(\kappa/\tilde{\kappa}) + G(\tilde{\kappa}) - G(\kappa). \end{aligned} \tag{58}$$

Hence we obtain the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ of the linear dissipative bosonic channel

$$I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = G(\kappa) - G(\kappa/\tilde{\kappa}). \tag{59}$$

On the other hand, the entropy exchange $S_e(\hat{\rho}, \hat{\mathcal{L}})$ and the coherent information $I_C(\hat{\rho}, \hat{\mathcal{L}})$ of the linear dissipative channel are calculated to be

$$S_e(\hat{\rho}, \hat{\mathcal{L}}) = G(\kappa/\tilde{\kappa}) \tag{60}$$

$$I_C(\hat{\rho}, \hat{\mathcal{L}}) = G(\tilde{\kappa}) - G(\kappa/\tilde{\kappa}). \tag{61}$$

In figure 3, the Ψ-information (59) and the coherent information (61) are plotted as functions of $\bar{n} = |\alpha|^2$. When the dissipation of the quantum channel is sufficiently large ($\tilde{\kappa} \approx 1$), we obtain $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 0$ and $I_C(\hat{\rho}, \hat{\mathcal{L}}) = -S(\hat{\rho})$. If the average photon number $\bar{n}$ is very large, the equality $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = I_C(\hat{\rho}, \hat{\mathcal{L}}) = 0$ is established. The entanglement fidelity $F_e(\hat{\rho}, \hat{\mathcal{L}})$ and

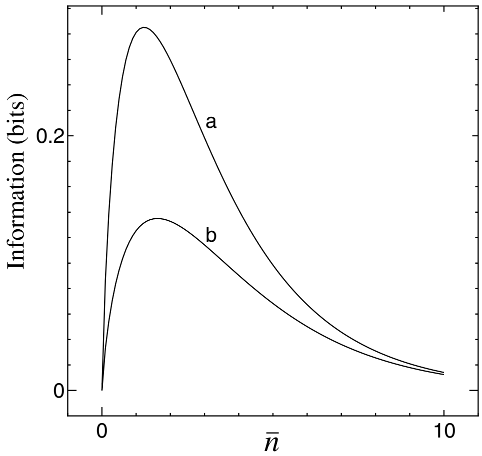

Figure 3. The Ψ-information (a) and the coherent information (b) of the linear dissipative channel with $g = 0.5$. In the figure, $\bar{n} = |\alpha|^2$ and the information is measured in bits.

the Ψ-fidelity $F_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ are given by

$$F_e(\hat{\rho}, \hat{\mathcal{L}}) = \frac{1}{4}\left[1 + \mathrm{e}^{-\bar{n}(1-\mathrm{e}^{-\frac{1}{2}g})^2} + 2\mathrm{e}^{-\bar{n}(1-\mathrm{e}^{-\frac{1}{2}g})}\right] \tag{62}$$

$$F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = F_e(\hat{\rho}, \hat{\mathcal{L}}) + \frac{\mathrm{e}^{-\bar{n}\mathrm{e}^{-g}}}{4(1-\mathrm{e}^{-\bar{n}})}\left[1 - \mathrm{e}^{-\bar{n}(1-\mathrm{e}^{-\frac{1}{2}g})}\right]^2 \tag{63}$$

which satisfy $\lim_{\bar{n}\to 0} F_e(\hat{\rho}, \hat{\mathcal{L}}) = \lim_{\bar{n}\to 0} F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = 1$. This is consistent with the fact that the vacuum state $|0\rangle$ remains unchanged in the linear dissipative channel (50).

5. Summary

In this paper, we have considered the partial reversibility of quantum operations. For this purpose, we have introduced the information-theoretic quantities, the Ψ-entropy and the Ψ-information, of quantum operations. For the classical-like quantum channel, the Ψ-entropy and the Ψ-information become the conditional entropy and the Shannon mutual information. The necessary and sufficient condition for a quantum operation to be partially reversible is that the Ψ-information of the quantum operation is equal to the von Neumann entropy of the input quantum state. It is important to note that the complete reversibility of quantum operations means that the coherent information is equal to the von Neumann entropy of the input quantum state. Completely reversible operations are partially reversible. We have calculated the Ψ-information for the quantum depolarizing channel of a qubit and the linear dissipative channel of a single-mode bosonic system. We have compared the results with those obtained for the coherent information. In this paper, we have confined ourselves to considering quantum operations for quantum states defined on a finite-dimensional Hilbert space. The coherent information and the entropy exchange that appeared in the complete reversibility of quantum operations are also restricted to a finite-dimensional Hilbert space [8, 9]. Continuous variable states such as a Gaussian quantum state are very important in quantum information processing [18–20]. Thus it is very important to generalize the complete reversibility and partial reversibility of quantum operation for an infinite-dimensional Hilbert space. In this generalization, an operator-algebraic approach may play an important role [17, 21, 22]. However, such a generalization is not trivial and needs further consideration.

Appendix A. Proof of proposition 1

For a quantum operation $\hat{\mathcal{L}}$ with the operator-sum representation $\hat{\mathcal{L}}\hat{X} = \sum_\mu \hat{A}_\mu \hat{X} \hat{A}_\mu^\dagger$ and a quantum state $\hat{\rho}$ with the spectral decomposition $\hat{\rho} = \sum_{k=1}^N \pi_k |\psi_k\rangle\langle\psi_k| \in \mathcal{Q}(\Psi)$, the entanglement fidelity $F_e(\hat{\rho}, \hat{\mathcal{L}})$ and the Ψ-fidelity $F_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ can be expressed as

$$F_e(\hat{\rho}, \hat{\mathcal{L}}) = \sum_\mu \left| \sum_{k=1}^N \pi_k \langle\psi_k|\hat{A}_\mu|\psi_k\rangle \right|^2 \tag{A.1}$$

$$F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = \sum_\mu \sum_{k=1}^N \pi_k |\langle\psi_k|\hat{A}_\mu|\psi_k\rangle|^2. \tag{A.2}$$

Using the Schwarz inequality and the normalization condition $\sum_{k=1}^N \pi_k = 1$, we obtain

$$\begin{aligned}\sum_{k=1}^N \pi_k |\langle\psi_k|\hat{A}_\mu|\psi_k\rangle|^2 &= \sum_{k=1}^N |\sqrt{\pi_k}\langle\psi_k|\hat{A}_\mu|\psi_k\rangle|^2 \sum_{k=1}^N \left(\sqrt{\pi_k}\right)^2 \\ &\geqslant \left| \sum_{k=1}^N \sqrt{\pi_k} |\sqrt{\pi_k}\langle\psi_k|\hat{A}_\mu|\psi_k\rangle| \right|^2 \\ &= \left| \sum_{k=1}^N \pi_k |\langle\psi_k|\hat{A}_\mu|\psi_k\rangle| \right|^2 \\ &\geqslant \left| \sum_{k=1}^N \pi_k \langle\psi_k|\hat{A}_\mu|\psi_k\rangle \right|^2.\end{aligned} \tag{A.3}$$

Taking the summation with respect to μ, we obtain inequality (22) from equations (A.1) and (A.2).

Appendix B. Proof of proposition 2

The Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ can be expressed in terms of the quantum relative entropy

$$\begin{aligned}I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) &= S(\hat{\mathcal{L}}\hat{\rho}) - \sum_{k=1}^N \pi_k S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|) \\ &= \sum_{k=1}^N \pi_k S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k| : \hat{\mathcal{L}}\hat{\rho})\end{aligned} \tag{B.1}$$

where $\hat{\rho} = \sum_{k=1}^N \pi_k |\psi_k\rangle\langle\psi_k|$ and $S(\hat{\rho} : \hat{\sigma}) = \mathrm{Tr}[\hat{\rho}(\log\hat{\rho} - \log\hat{\sigma})]$ is the quantum relative entropy [17]. Since for any trace-preserving quantum operation $\hat{\mathcal{L}}$, the quantum relative entropy satisfies the inequality $S(\hat{\rho} : \hat{\sigma}) \geqslant S(\hat{\mathcal{L}}\hat{\rho} : \hat{\mathcal{L}}\hat{\sigma})$, we obtain

$$I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \leqslant \sum_{k=1}^N \pi_k S(|\psi_k\rangle\langle\psi_k| : \hat{\rho}) = S(\hat{\rho}). \tag{B.2}$$

On the other hand, since the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ is non-negative, the inequality $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \leqslant S(\hat{\mathcal{L}}\hat{\rho})$ is trivial from definition (24). Hence we have shown inequality (26). In the same way, using the inequality $S(\hat{\rho} : \hat{\sigma}) \geqslant S(\hat{\mathcal{L}}\hat{\rho} : \hat{\mathcal{L}}\hat{\sigma})$, we can derive the quantum data processing inequality (27) of the Ψ-information $I_\Psi(\hat{\rho}, \hat{\mathcal{L}})$

$$\begin{aligned} I_\Psi(\hat{\rho}, \hat{\mathcal{K}}\hat{\mathcal{L}}) &= \sum_{k=1}^{N} \pi_k S(\hat{\mathcal{K}}\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k| : \hat{\mathcal{K}}\hat{\mathcal{L}}\hat{\rho}) \\ &\leqslant \sum_{k=1}^{N} \pi_k S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k| : \hat{\mathcal{L}}\hat{\rho}) \\ &= I_\Psi(\hat{\rho}, \hat{\mathcal{L}}). \end{aligned} \tag{B.3}$$

To prove the quantum Fano inequality of the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$, we introduce a non-negative parameter

$$F_k(l) = \langle\psi_l|(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|)|\psi_l\rangle \tag{B.4}$$

in terms of which the Ψ-fidelity is expressed as $F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = \sum_{k=1}^{N} \pi_k F_k(k)$. Then we obtain

$$\begin{aligned} S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|) &= -\operatorname{Tr}[(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|) \log(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|)] \\ &\leqslant -\sum_{l=1}^{N} F_k(l) \log F_k(l) \\ &= -F_k(k) \log F_k(k) - \sum_{\substack{l=1 \\ (l\neq k)}}^{N} F_k(l) \log F_k(l). \end{aligned} \tag{B.5}$$

Here we define a probability distribution $G_k(l) = F_k(l)/[1 - F_k(k)]$ $(l \neq k)$ which is normalized as $\sum_{l=1(l\neq k)}^{N} G_k(l) = 1$. Then we have

$$\begin{aligned} S(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|) &\leqslant H[F_k(k)] - [1 - F_k(l)] \sum_{\substack{l=1 \\ (l\neq k)}}^{N} G_k(l) \log G_k(l) \\ &\leqslant H[F_k(k)] + [1 - F_k(l)] \log(N - 1) \end{aligned} \tag{B.6}$$

where $H(x) = -x \log x - (1-x) \log(1-x)$. In the second inequality, we have used the fact that $-\sum_{l=1(l\neq k)}^{N} G_k(l) \log G_k(l) \leqslant \log(N-1)$. When we average inequality (B.6) with the probability π_k and use the equality $F_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = \sum_{k=1}^{N} \pi_k F_k(k)$, we can obtain the quantum Fano inequality (28).

Appendix C. Proof of theorems 1 and 2

We first prove that equality (29) is necessary for a quantum operation $\hat{\mathcal{L}}$ to be partially reversible. We assume that there is a completely positive map $\hat{\mathcal{R}}$ such that $F_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 1$ for a quantum state $\hat{\rho}$ whose eigenstates belong to the complete orthonormal system Ψ. Then the quantum Fano inequality (28) implies $S_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 0$. Using equations (26) and (27), we obtain

$$\begin{aligned} S(\hat{\rho}) &\geqslant I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \\ &\geqslant I_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) \\ &= S(\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho}) - S_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) \\ &= S(\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho}) \\ &= S(\hat{\rho}) \end{aligned} \tag{C.1}$$

where we have used the fact that $\hat{\mathcal{R}}\hat{\mathcal{L}}\hat{\rho} = \hat{\rho}$ for $\hat{\rho} = \sum_k \pi_k |\psi_k\rangle\langle\psi_k|$ $(|\psi_k\rangle \in \Psi)$. Hence the partial reversibility of the quantum operation $\hat{\mathcal{L}}$ yields equality (29).

Next we prove that equality (29) is sufficient for the partial reversibility of a quantum operation $\hat{\mathcal{L}}$. To do this, we explicitly construct a quantum operation $\hat{\mathcal{R}}$ such that $F_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 1$ when equality (29) holds. We introduce an auxiliary Hilbert space $\mathcal{H}'$ which has a complete orthonormal system $\{|\phi_\mu\rangle\}$ and define a quantum state $|\Psi_j\rangle$ of the Hilbert space $\mathcal{H}_N \otimes \mathcal{H}'$

$$|\Psi_j\rangle = \sum_\mu \hat{A}_\mu |\psi_j\rangle \otimes |\phi_\mu\rangle \tag{C.2}$$

where $|\psi_j\rangle \in \Psi$ is the eigenstate of the quantum state $\hat{\rho}$ and $\hat{\mathcal{L}}\hat{X} = \sum_\mu \hat{A}_\mu \hat{X} \hat{A}_\mu^\dagger$. Since $\sum_\mu \hat{A}_\mu^\dagger \hat{A}_\mu = \hat{1}$ is satisfied, the quantum state $|\Psi_j\rangle$ is normalized. We denote as $\hat{\rho}_j$ and $\hat{\rho}'_j$ the reduced quantum states defined on the Hilbert spaces $\mathcal{H}_N$ and $\mathcal{H}'$. Then the equality $S(\hat{\rho}_j) = S(\hat{\rho}'_j)$ holds due to the Araki–Lieb inequality and to the fact that $|\Psi\rangle\langle\Psi|$ when reduced to the Hilbert space $\mathcal{H}'$ amounts to $\hat{\mathcal{L}}\hat{\rho}$. The reduced quantum state $\hat{\rho}'_j$ can be expressed as

$$\hat{\rho}'_j = \sum_\mu \sum_\nu W^j_{\mu\nu} |\phi_\mu\rangle\langle\phi_\nu| \qquad W^j_{\mu\nu} = \langle\psi_j|\hat{A}_\nu^\dagger \hat{A}_\mu|\psi_j\rangle \tag{C.3}$$

which implies that the eigenvalues of the reduced quantum state $\hat{\rho}'_j$ are equal to those of the matrix $\mathbf{W}^j = \left(W^j_{\mu\nu}\right)$. Hence we obtain

$$S(\hat{\rho}_j) = S(\hat{\rho}'_j) = -\mathrm{Tr}[\mathbf{W}^j \log \mathbf{W}^j]. \tag{C.4}$$

Using $\hat{\rho}_j = \hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|$, we can express the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ as

$$S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = \sum_{j=1}^N \pi_j S(\hat{\rho}_j) = -\sum_{j=1}^N \pi_j \mathrm{Tr}[\mathbf{W}^j \log \mathbf{W}^j]. \tag{C.5}$$

To obtain the matrix form of the von Neumann entropy of the quantum state $\hat{\mathcal{L}}\hat{\rho}$, we introduce an auxiliary Hilbert space $\mathcal{H}'$ which has a complete orthonormal system $\{|\phi_{k\mu}\rangle\}$, and we define a normalized state vector $|\Psi\rangle$ of the Hilbert space $\mathcal{H}_N \otimes \mathcal{H}'$

$$|\Psi\rangle = \sum_\mu \sum_{k=1}^N \sqrt{\pi_k} \hat{A}_\mu |\psi_k\rangle \otimes |\phi_{k\mu}\rangle. \tag{C.6}$$

Then the reduced quantum state $\hat{\rho}'$ of the Hilbert space $\mathcal{H}'$ is given by

$$\hat{\rho}' = \sum_\mu \sum_\nu \sum_{j=1}^N \sum_{k=1}^N W^{jk}_{\mu\nu} |\phi_{j\mu}\rangle\langle\phi_{k\nu}| \qquad W^{jk}_{\mu\nu} = \sqrt{\pi_j \pi_k} \langle\psi_k|\hat{A}_\nu^\dagger \hat{A}_\mu|\psi_j\rangle. \tag{C.7}$$

Since the eigenvalues of the reduced quantum state $\hat{\rho}'$ are equal to those of the matrix and the equality $S(\hat{\mathcal{L}}\hat{\rho}) = S(\hat{\rho}')$ holds due to the Araki–Lieb inequality, we can find that the von Neumann entropy of the quantum state $\hat{\mathcal{L}}\hat{\rho}$ is expressed as

$$S(\hat{\mathcal{L}}\hat{\rho}) = -\,\mathrm{Tr}[\tilde{\mathbf{W}} \log \tilde{\mathbf{W}}] \tag{C.8}$$

where $\tilde{\mathbf{W}} = \left(W^{jk}_{\mu\nu}\right)$. Using the relation $W^{jj}_{\mu\nu} = \pi_j W^j_{\mu\nu}$, we obtain

$$\begin{aligned} S(\hat{\mathcal{L}}\hat{\rho}) &= -\operatorname{Tr}[\tilde{\mathbf{W}} \log \tilde{\mathbf{W}}] \\ &\leqslant -\sum_{j=1}^{N} \operatorname{Tr}[\tilde{\mathbf{W}}^{jj} \log \tilde{\mathbf{W}}^{jj}] \\ &= -\sum_{j=1}^{N} \operatorname{Tr}[(\pi_j \mathbf{W}^j) \log(\pi_j \mathbf{W}^j)] \\ &= -\sum_{j=1}^{N} \pi_j \log \pi_j - \sum_{j=1}^{N} \pi_j \operatorname{Tr}[\mathbf{W}^j \log \mathbf{W}^j] \\ &= S(\hat{\rho}) + S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) \end{aligned} \tag{C.9}$$

where $\tilde{\mathbf{W}}^{jj}$ is the matrix with the element $W^{jj}_{\mu\nu}$. When the condition $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ is satisfied, the equality must be attained on the second line of this equation. This implies that the matrix $\tilde{\mathbf{W}} = \left(W^{jk}_{\mu\nu}\right)$ must be diagonal with respect to the upper indices. Thus we have found that the condition $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ is equivalent to

$$\langle\psi_k|\hat{A}^\dagger_\nu \hat{A}_\mu|\psi_j\rangle = 0 \qquad (j \neq k). \tag{C.10}$$

We assume that the quantum operation $\hat{\mathcal{L}}$ satisfies the condition $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ or equivalently $\langle\psi_k|\hat{A}^\dagger_\nu \hat{A}_\mu|\psi_j\rangle = 0$ $(j \neq k)$. Then for $j \neq k$, we obtain

$$\begin{aligned} \operatorname{Tr}[(\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|)^\dagger(\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|)] &= \sum_\mu \sum_\nu \operatorname{Tr}\left[\left(\hat{A}_\nu|\psi_j\rangle\langle\psi_j|\hat{A}^\dagger_\nu\right)\left(\hat{A}_\mu|\psi_k\rangle\langle\psi_k|\hat{A}^\dagger_\mu\right)\right] \\ &= \sum_\mu \sum_\nu |\langle\psi_j|\hat{A}^\dagger_\nu \hat{A}_\mu|\psi_k\rangle|^2 \\ &= 0 \end{aligned} \tag{C.11}$$

which means that the operator $\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|$ is orthogonal to $\hat{\mathcal{L}}|\psi_k\rangle\langle\psi_k|$ $(j \neq k)$ with respect to the Hilbert–Schmidt product. The spectral decomposition of the operator $\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|$ is given by

$$\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j| = \sum_\xi \lambda_{j\xi}|\phi_{j\xi}\rangle\langle\phi_{j\xi}| \qquad \left(\sum_\xi \lambda_{j\xi} = 1, \quad \lambda_{j\xi} \geqslant 0\right). \tag{C.12}$$

Because of the orthogonality of the operator $\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|$, the eigenstate $|\phi_{j\xi}\rangle$ satisfies the relation

$$\langle\phi_{j\xi}|\phi_{k\zeta}\rangle = \delta_{jk}\delta_{\xi\zeta} \qquad \sum_\xi |\phi_{j\xi}\rangle\langle\phi_{j\xi}| = \tilde{1}_j \tag{C.13}$$

where $\tilde{1}_j$ is the identity operator defined on the support space $\tilde{\mathcal{H}}_j$ of the operator $\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|$. Using the orthonormal vectors $|\psi_j\rangle$ and $|\phi_{j\xi}\rangle$, we introduce

$$\hat{B}_\xi = \sum_j |\psi_j\rangle\langle\phi_{j\xi}| \tag{C.14}$$

which satisfies the relation $\sum_\xi \hat{B}^\dagger_\xi \hat{B}_\xi = \tilde{1}$, where $\tilde{1}$ is the identity defined on the support space of the quantum state $\hat{\mathcal{L}}\hat{\rho}$. Then the quantum operation $\hat{\mathcal{R}}$ defined below is a trace-preserving completely positive map

$$\hat{\mathcal{R}}\hat{X} = \sum_\xi \hat{B}_\xi \hat{X} \hat{B}^\dagger_\xi. \tag{C.15}$$

It is found from equation (C.12) that the quantum operation $\hat{\mathcal{R}}$ satisfies

$$\begin{aligned}\hat{\mathcal{R}}(\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|) &= \sum_\xi \hat{B}_\xi(\hat{\mathcal{L}}|\psi_j\rangle\langle\psi_j|)\hat{B}_\xi^\dagger \\ &= \sum_\xi \left(\sum_k |\psi_k\rangle\langle\phi_{k\xi}| \sum_\zeta \lambda_{j\zeta}|\phi_{j\zeta}\rangle\langle\phi_{j\zeta}| \sum_l |\phi_{l\xi}\rangle\langle\psi_l|\right) \\ &= \sum_\xi \lambda_{j\xi}|\psi_j\rangle\langle\psi_j| \\ &= |\psi_j\rangle\langle\psi_j|. \end{aligned} \tag{C.16}$$

This result is equivalent to the equality $F_\Psi(\hat{\rho}, \hat{\mathcal{R}}\hat{\mathcal{L}}) = 1$. Therefore, it has been found that the equality $I_\Psi(\hat{\rho}, \hat{\mathcal{L}}) = S(\hat{\rho})$ implies the partial reversibility of the quantum operation $\hat{\mathcal{L}}$. Furthermore, it is found from equation (C.10) that condition (30) is also necessary and sufficient for a quantum operation $\hat{\mathcal{L}}$ to be partially reversible.

Appendix D. Proof of proposition 3

The entropy exchange $S_e(\hat{\rho}, \hat{\mathcal{L}})$ of a quantum operation $\hat{\mathcal{L}}$ which has the operator-sum representation $\hat{\mathcal{L}}\hat{X} = \sum_\mu \hat{A}_\mu \hat{X}\hat{A}_\mu^\dagger$ can be expressed as

$$S_e(\hat{\rho}, \hat{\mathcal{L}}) = -\operatorname{Tr}[\mathbf{W}\log\mathbf{W}] \tag{D.1}$$

where $\mathbf{W}$ is the matrix whose element is given by $W_{\mu\nu} = \operatorname{Tr}\left[\hat{A}_\nu^\dagger \hat{A}_\mu \hat{\rho}\right]$ [8]. The matrix $\mathbf{W}$ is the average of the matrix $\mathbf{W}^j$ with the probability π_j

$$\begin{aligned}\sum_j \pi_j W_{\mu\nu}^j &= \sum_j \pi_j \langle\psi_j|\hat{A}_\nu^\dagger \hat{A}_\mu|\psi_j\rangle \\ &= \sum_j \pi_j \operatorname{Tr}\left(\hat{A}_\nu^\dagger \hat{A}_\mu |\psi_j\rangle\langle\psi_j|\right) \\ &= \operatorname{Tr}\left[\hat{A}_\nu^\dagger \hat{A}_\mu \hat{\rho}\right] \\ &= W_{\mu\nu}. \end{aligned} \tag{D.2}$$

Using the concavity of the entropic function, we obtain

$$\begin{aligned}S_\Psi(\hat{\rho}, \hat{\mathcal{L}}) &= \sum_j \pi_j S(\hat{\rho}_j) \\ &= -\sum_j \pi_j \operatorname{Tr}[\mathbf{W}^j \log \mathbf{W}^j] \\ &\leqslant -\operatorname{Tr}[\mathbf{W}\log\mathbf{W}] \\ &= S_s(\hat{\rho}, \hat{\mathcal{L}}). \end{aligned} \tag{D.3}$$

Thus the Ψ-entropy $S_\Psi(\hat{\rho}, \hat{\mathcal{L}})$ is not greater than the entropy exchange $S_e(\hat{\rho}, \hat{\mathcal{L}})$. This result implies inequality (31).

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